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Non-Markovian Reversible Chapman–Kolmogorov Measures on Subshifts of Finite Type

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We consider shift-invariant probability measures on subshift dynamical systems with a transition matrix A which satisfies the Chapman–Kolmogorov equation for some stochastic matrix Π compatible with A. We call them Chapman–Kolmogorov measures. A nonequilibrium entropy is associated to this class of dynamical systems. We show that if A is irreducible and aperiodic, then there are Chapman–Kolmogorov measures distinct from the Markov chain associated with Π and its invariant row probability vector q. If, moreover, (q, Π) is a reversible chain, then we construct reversible Chapman–Kolmogorov measures on the subshift which are distinct from (q, Π).

KEY WORDS: Topological Markov chain; Chapman–Kolmogorov equation; stationary non-Markovian stochastic processes; detailed balance.

INTRODUCTION

The statistical foundations of nonequilibrium thermodynamics is based on the hypothesis that the macroscopic observables are represented by random processes which obey the Chapman–Kolmogorov equation (see, e.g., ref. 6, Chapter VII). The dynamical justification of this hypothesis is based on a coarse-graining procedure which goes back to Gibbs and has been largely discussed by Ehrenfest and Ehrenfest,[7] Uhlenbeck,[9] and Kac.[8] To formulate this procedure in an abstract way, we consider a conservative dynamical system given by a one-to-one measurable transformation S acting on a probability space $(\Omega, \mathscr{A})$ and preserving the probability measure μ. It can be shown[3] that such a coarse-graining procedure is equivalent to the construction of a partition of Ω which we call the Chapman–Kolmogorov

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partition. A finite partition $\mathscr{P} = \{P_0, ..., P_{k-1}\}$ of Ω, where $P_i \in \mathscr{A}$ and $\mu(P_i) > 0$ for all i, is a Chapman–Kolmogorov partition if the family of matrices Π_n given by

$$(\Pi_n)_{ij} = \mu(S^{-n}P_j \mid P_i) \tag{1}$$

forms a semigroup: $\Pi_{n+n'} = \Pi_n \Pi_{n'}$, n, $n' \geqslant 0$; that is, $\Pi_n = \Pi^n$, where Π denotes Π_1.

If one, moreover, requires the entropy functional of the non-equilibrium distributions to approach its equilibrium value, then Π should be 0 irreducible and aperiodic [3]. It is generally supposed that the stationary process associated to the dynamical system and the partition $\mathscr{P}$ is a Markov process; this leads to the Chapman–Kolmogorov property (see, e.g., ref. 6). Examples of non-Markovian stationary processes satisfying the Chapman–Kolmogorov equation have been obtained by P. Lévy. In ref. 4 we constructed a family of processes with infinite memory satisfying this equation for positive Π. In some cases, the Markov chain is the only process satisfying the Chapman–Kolmogorov equation (for example, if Π is the identity matrix). In general, the processes satisfying the Chapman–Kolmogorov property are more likely to be non-Markovian, as has been shown in ref. 5. Here we construct many distinct stationary processes satisfying the Chapman–Kolmogorov equation for irreducible and aperiodic Π.

The construction of such partitions is also related to a problem in the spectral theory of dynamical systems (see, e.g., ref. 2) on the realization of a given spectral type with functions having simple spatial properties. Alexeyev[1] has shown that the maximal spectral type may be realized by a bounded function. It can be noticed that the existence of a Chapman–Kolmogorov partition with irreducible and aperiodic matrix Π implies that the system has a Lebesgue spectral type realized by a function taking a finite number of values.

The existence of a Chapman–Kolmogorov partition is constraining for the dynamical system. In fact, if we consider the case of a partition with two cells, with Π irreducible and aperiodic, Π has two real eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \lambda$ such that $|\lambda| < 1$. The function χ given by

$$\chi = \alpha_0 1_{P_0} + \alpha_1 1_{P_1}$$

where (α_0, α_1) is an eigenvector of Π corresponding to λ_2, has a spectral measure μ_χ eiquivalent to the Lebesgue measure. In fact, one verifies that χ is orthogonal to 1 in L^2_μ, and if we take $\|\chi\| = 1$, then we obtain by simple computations

$$1_{P_0} = \mu_0(1 + \alpha_0 \chi)$$

$$\langle \chi, U^n \chi \rangle = \lambda^n \|\chi\|^2 = \lambda^{|n|}$$

where $(Uf)(X) = f(S^{-1}X)$ and $\mu_i = \mu(P_i)$. Then we get

$$\langle 1_{P_0}, U^n 1_{P_0} \rangle = \langle \mu_0(1+\alpha_0\chi), \mu_0(1+\alpha_0 U^n\chi) \rangle$$
$$= \mu_0^2(1+\alpha_0^2\lambda^n) \tag{2}$$

Now, α_0 and α_1 are computed from the orthonormality of $(1, \chi)$, $\alpha_0^2 = \mu_1/\mu_0$, $\alpha_1^2 = \mu_0/\mu_1$. This gives the following condition:

$$\mu(P_0 \cap S^{-n}P_0) - \mu(P_0)^2 = \mu(P_0)\,\mu(P_1)\,\lambda^n \tag{3}$$

Conversely, if P_0 is a subset of Ω, such that $0 < \mu(P_0) < 1$, and satisfying (3) for some λ, then the partition $P = \{P_0, P_1 = P_0^c\}$ is a Chapman–Kolmogorov partition. This is also equivalent to

$$\mu(P_i \cap S^{-n}P_j) - \mu(P_i)\,\mu(P_j) = \alpha_i\alpha_j\mu_i\mu_j\lambda^n \tag{4}$$

$i = 0$ or 1, for any n.

This raises the question of whether this condition may be realized for any dynamical system with Lebesgue spectral component, in particular, for those systems with zero entropy.

In ref. 5 we gave examples of ergodic systems with *zero entropy* having partitions $\mathscr{P}$ which are independent at different times.

In Section 1 we construct non-Markov measures for topological Markov chains (subshifts) having irreducible and aperiodic transition matrix. These measures, which may be ergodic, can be used as invariant measures for hyperbolic attractors.

In Section 2 we consider the problem of construction of *reversible* stationary stochastic processes from reversible Chapman–Kolmogorov partitions. This corresponds to the so-called "microscopic reversibility" used in the statistical theory of nonequilibrium thermodynamics in order to justify the Onsager relations.[6] This can be formulated as follows: The conservative dynamical system $(\Omega, \mathscr{A}, S, \mu)$ is called reversible if there exists a one-to-one μ-preserving transformation I such that $I^2 = 1$ and $ISI = S$. Moreover, we suppose that there exists a partition $\mathscr{P}$ such that $IP_i = P_i$ for all i. This implies that

$$\mu(P_{x_0} \cap T^{-1}P_{x_1} \cap \cdots \cap T^{-n}P_{x_n}) = \mu(P_{x_n} \cap T^{-1}P_{x_{n-1}} \cap \cdots \cap T^{-n}P_{x_0}) \tag{5}$$

where $x_i = 0,..., k-1$. Let Φ be the mapping: $\Omega \to K^{\mathbb{Z}}$, $\phi(\omega) = (x_i)$ defined by: $T^n\omega \in P_{x_n}$. This mapping transports μ into $\Phi\mu$, which is shift invariant on $K^{\mathbb{Z}}$, such that

$$\phi\mu(\omega_0 = x_0,..., \omega_n = x_n) = \phi\mu(\omega_0 = x_n,..., \omega_n = x_0) \tag{6}$$

If, moreover, $\mathscr{P}$ is a Chapman–Kolmogorov partition, then the reversibility property (5) implies that

$$\mu(P_i)(\Pi^n)_{ij} = \mu(P_j)(\Pi^n)_{ji} \tag{7}$$

This is called the detailed balance equation (also "microscopic reversibility"). We show that the Chapman–Kolmogorov equation and the detailed balance equation may be satisfied for stationary stochastic processes which are not Markovian and for dynamical systems even non-ergodic. Here also we see that the Markovian character is not necessary.

1. NON-MARKOVIAN INVARIANT MEASURE ON SUBSHIFT SYSTEMS

Let $K = \{0, 1,..., k-1\}$, $k \geqslant 2$, and let A be a $k \times k$ matrix whose elements A_{ij} are zero or one. Let Ω_A be the set of all doubly infinite sequences (ω_i), $\omega_i \in K$, such that $A_{\omega_i, \omega_{i+1}} = 1$. Let σ be the shift transformation: $(\sigma\omega)_i = \omega_{i+1}$. Such a system is called a subshift or a topological Markov chain.

Let Π be a stochastic $k \times k$ matrix compatible with A, that is, $\Pi_{ij} > 0$ if and only if $A_{ij} > 0$. A p-uplet $\tau = (i_1,..., i_p)$ such that $A_{i_1, i_2} = \cdots = A_{i_{p-1}, i_p} = 1$ will be called an admissible word of length $|\tau| = p$. Let $q = (p_i)$ be a row probability vector invariant under Π. The Markov measure μ_π is defined by

$$\mu_\pi(\omega_n = x_0,..., \omega_{n+p} = x_p) = p_{x_0} \Pi_{x_0, x_1} \cdots \Pi_{x_{p-1}, x_p} \tag{8}$$

which we simply denote $\mu_\pi(x_0,..., x_p)$. Thus μ_π is defined on the σ-algebra $\mathscr{A}_A$ generated by the *admissible* cylindrical sets

$$\{\omega: \omega_n = i_1, \omega_{n+1} = i_2,..., \omega_{n+p} = i_p\}$$

where $\tau = (i_1,..., i_p)$ is an admissible word.

Definition. A σ-invariant probability measure on $(\Omega_A, \mathscr{A}_A)$ that satisfies the Chapman–Kolmogorov equation

$$\nu(\omega_m = j \mid \omega_0 = i) = (\Pi^m)_{ij} \tag{9}$$

for $m > 0$, will be called a Chapman–Kolmogorov measure for the matrix Π. We denote by $C_{\pi, A}$ the set of these measures.

Let $C_{\pi, n, A}$ be the set of all probability measures μ on K^{n+1} charging the admissible cylindrical sets, invariant under the left-shift and satisfying (9) for $m = 1,..., n$. For any such μ, we define a measure ν_0 on $(\Omega_A, \mathscr{A}_A)$ by

$$\nu_0(\{\omega: \omega_0 = x_0, \omega_1 = x_1,..., \omega_{rn} = x_{rn}\})$$
$$= \mu(x_0,..., x_n)\, \mu(x_{n+1},..., x_{2n} \mid x_n) \cdots \mu(x_{(r-1)n+1},..., x_{rn} \mid x_{(r-1)n}) \tag{10}$$

where we denote

$$\mu(\{\omega: \omega_0 = x_0,..., \omega_k = x_k\}) = \mu(x_0,..., x_k), \qquad k \leqslant n$$

$$\mu(\{\omega: \omega_1 = y_1,..., \omega_n = y_n \mid \omega_0 = y_0\}) = \mu(y_1,..., y_n \mid y_0)$$

If μ is distinct from $\mu_\pi \mid_{K^{n+1}}$, then $\varphi_n(\mu) = (1/n) \sum_{i=0}^{n-1} \sigma^i v_0$ is in $C_{\pi,A}$ and it is distinct from μ_π.[5]

Theorem 1. If Π is irreducible and aperiodic, then there is N such that, for $n \geqslant N$, $C_{\pi,n,A}$ contains measures distinct from μ_π.

Proof. In order to construct such μ, we shall solve the system

$$\sum_j \mu(i\tau j) = \mu_\pi(i\tau) \tag{11}$$

$$\sum_i \mu(i\tau j) = \mu_\pi(\tau j) \tag{12}$$

$$\sum_\tau \mu(i\tau j) = \mu_\pi(\omega_0 = i, \omega_{n+1} = j) \tag{13}$$

where $\tau = (\tau_1,..., \tau_n)$ is an admissible word.

For any fixed τ, Eqs. (11) and (12) form a linear system of equations denoted $AX = Y^0$, where X represents the variables $\mu(i\tau j)$. As $\mu_\pi(i\tau j)$ is a solution of the system (which we denote X^0), the general solution of the system is $X^0 + Z$, where Z is the general solution to

$$AZ = 0 \tag{14}$$

By ordering the variables $X = \{\mu(i\tau j)\}$ and $Z = \{z_{ij}(\tau)\}$ lexicographically $\{(0, 0), (0, 1),..., (0, k-1), (1, 0),...\}$, the matrix A takes the form

$$
\begin{matrix} k\left\{\begin{matrix}\\ \\ \\ \\ \\ \\ \end{matrix}\right. \\ k\left\{\begin{matrix}\\ \\ \\ \\ \\ \\ \end{matrix}\right. \end{matrix}
\underbrace{\left[\begin{array}{cccc|cccc|cc|ccc}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & & 0 & 0 & \cdots & 0 \\
\cdot & & & \cdot & 0 & 0 & \cdots & 0 & \cdots & \cdot & & & \cdot \\
\cdot & & & \cdot & \cdot & & & \cdot & & \cdot & & & \cdot \\
\cdot & & & \cdot & \cdot & & & \cdot & & \cdot & & & \cdot \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & & 1 & 1 & \cdots & 1 \\
\hline
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & & 1 & 0 & \cdots & 0 \\
0 & 1 & & \cdot & 0 & 1 & & \cdot & & 0 & 1 & & \cdot \\
\cdot & \cdot & & \cdot & \cdot & & & \cdot & \cdots & \cdot & & & \cdot \\
\cdot & & & \cdot & \cdot & & & \cdot & & \cdot & & & \cdot \\
\cdot & & & 0 & \cdot & & & 0 & & \cdot & & & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & & 0 & 0 & 0 & 1
\end{array}\right]}_{k \times k}
$$

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The rank of this matrix is $(2k-1)$. We take as principal variables for the system $AZ=0$ those indexed by $\{i, 0\}$, $i=0,..., k-2$, and $(k-1, j)$, $j=0,..., k-1$. They are expressed in terms of the other variables, which we call the parameters J. Taking, moreover, into account the positivity of $\mu(i\tau j)$, $Z \geqslant -X^0$, we obtain

$$\begin{aligned}
z_{i,j}(\tau) &\geqslant -\mu_\pi(i\tau j), \qquad (i, j) \in J \\
z_{i,0}(\tau) &= -\sum_{j=1}^{k-1} z_{i,j}(\tau) \geqslant -\mu_\pi(i\tau 0), \qquad i=0,..., k-2 \\
z_{k-1,0}(\tau) &= \sum_{i=0}^{k-2} \sum_{j=1}^{k-1} z_{ij}(\tau) \geqslant -\mu_\pi(k-1, \tau, 0) \\
z_{k-1,j}(\tau) &= -\sum_{i=0}^{k-2} z_{i,j} \geqslant -\mu_\pi(k-1, \tau, j), \qquad j=1,..., k-1
\end{aligned} \tag{15}$$

All these inequalities determine intervals of variation of the parameters. Moreover, the Chapman–Kolmogorov equation (13) becomes

$$\sum_\tau z_{i,j}(\tau) = 0 \tag{16}$$

We shall look for solutions such that all the parameters but four are vanishing. The nonvanishing parameters are denoted $z_{i_0 j_0}(\tau)$, $z_{i_0 j_0}(\sigma)$, $z_{i_0 j_1}(\tau)$, $z_{i_0 j_1}(\sigma)$, $i_0 \neq k-1$, $j_0 \neq j_1$, $j_0 \neq 0$, $j_1 \neq 0$, $\tau \neq \sigma$, $|\tau| = |\sigma|$, We find from (15) that $z_{k-1,0}(\tau)$, $z_{k-1,0}(\sigma)$, $z_{k-1,j_0}(\tau)$, and $z_{k-1,j_0}(\sigma)$ are also nonnull. Such a solution implies that the number of states of the chain is at least 3, i.e., $k \geqslant 3$. The case $k=2$ will be discussed separately. The system (15) is then reduced to

$$\begin{aligned}
z_{i_0,j_0}(\tau) &\geqslant -\mu_\pi(i_0 \tau j_0) \\
z_{i_0,j_1}(\tau) &\geqslant -\mu_\pi(i_0 \tau j_1) \\
-\mu_\pi(k-1, \tau, 0) \leqslant z_{i_0,j_0}(\tau) + z_{i_0,j_1}(\tau) &\leqslant \mu_\pi(i_0 \tau, 0) \\
z_{i_0,j_0}(\tau) &\leqslant \mu_\pi(k-1, \tau, j_0) \\
z_{i_0,j_1}(\tau) &\leqslant \mu_\pi(k-1, \tau, j_1)
\end{aligned} \tag{15'}$$

and similarly for σ.

The Chapman–Kolmogorov equations reduce to

$$\begin{aligned}
z_{i_0,j_0}(\tau) &= -z_{i_0,j_0}(\sigma) \\
z_{i_0,j_1}(\tau) &= -z_{i_0,j_1}(\sigma)
\end{aligned} \tag{17}$$

A study of the system (15′) for τ and σ shows that the inequalities

$$\mu_\pi(i_0\tau j_0)\,\mu_\pi(i_0\tau j_1)\,\mu_\pi(k-1,\tau,j_0)\,\mu_\pi(k-1,\tau,j_1)>0 \tag{18}$$

$$\mu_\pi(i_0\sigma j_0)\,\mu_\pi(i_0\sigma j_1)\,\mu_\pi(k-1,\sigma,j_0)\,\mu_\pi(k-1,\sigma,j_1)>0 \tag{19}$$

are sufficient conditions for the existence of a nonunique solution. Condition (18) leaves us the freedom to choose $(z_{i_0,j_0}(\tau), z_{i_0,j_1}(\tau))$ in the subset

$$I_1=\{(x,y)\in\mathbb{R}^2: x+y=0,\ -\mu_\pi(i_0\tau j_0)\leqslant x<0,\ 0<y\leqslant\mu_\pi(k-1,\tau,j_0)\}$$

Similarly, (19) is a sufficient condition which leaves us the freedom to choose $(z_{i_0,j_0}(\sigma), z_{i_0,j_1}(\sigma))$ in the subset

$$I_2=\{(x,y)\in\mathbb{R}^2: x+y=0,\ 0<x\leqslant\mu_\pi(k-1,\sigma,j_0),\ -\mu_\pi(i_0,\sigma,j_1)\leqslant y<0\}$$

As Π is irreducible and aperiodic, to satisfy (18) and (19), it is sufficient to find $s, t\in K$, such that

$$\Pi_{k-1,s}\Pi_{i_0,s}\Pi_{tj_0}\Pi_{tj_1}>0 \tag{20}$$

In fact, in this case, there exists an admissible word $\delta=(\delta_1,\ldots,\delta_n)$ sufficiently long connecting s and t so that

$$\mu_\pi(i_0,s,\delta,t,j_0)>0, \qquad \mu_\pi(i_0,s,\delta,t,j_1)>0$$
$$\mu_\pi(k-1,s,\delta,t,j_0)>0, \qquad \mu_\pi(k-1,s,\delta,t,j_1)>0$$

Now, we find $\sigma\neq\tau\equiv(s,\delta,t)$ and satisfying (19): let $n_0>0$ such that $\Pi^{n_0}>0$; then for $n\geqslant 3n_0$, let $\lambda_{n_0+1}\in K$ such that $\lambda_{n_0+1}\neq\delta_{n_0+1}$; then there exist an admissible word $(\lambda_1,\ldots,\lambda_{n_0})$ connecting s and λ_{n_0+1} and an admissible word $(\lambda_{n_0+2},\ldots,\lambda_n)$ connecting λ_{n_0+1} and t. Thus, $\sigma=(s,\lambda_1,\ldots,\lambda_n,t)\neq\tau$.

Let us now show that there exist $s, t\in K$ such that (20) is satisfied under the conditions

$$i_0\neq k-1, \qquad j_0\neq 0, \qquad j_1\neq 0, \qquad j_0\neq j_1 \tag{21}$$

After some permutation S of $(0,\ldots,k-1)$, denoting $S(k-1)=u$, $S(i_0)=v$, $S(0)=l$, the conditions (20) and (21) read: there exist u, v, j_0, j_1, l such that $u\neq v$, $u\neq l$, $j_0\neq j_1$, $j_0\neq l$, $j_1\neq l$, $\Pi_{us}\times\Pi_{vs}\times\Pi_{tj_0}\times\Pi_{tj_1}>0$.

They can be satisfied if, in particular, we may choose $v=l$. Thus we need to prove the following result:

Lemma 1. Let Π be an irreducible and aperiodic $k\times k$ matrix, $k\geqslant 3$. Then there exist states l, u, j_0, j_1, s, t with $l\neq u$, $j_0\neq j_1$, $j_0\neq l$, $j_1\neq l$, such that

$$\Pi_{us}\times\Pi_{ls}\times\Pi_{tj_0}\times\Pi_{tj_1}>0 \tag{20′}$$

Proof. There exist t, j_0, j_1, $j_0 \neq j_1$, such that $\Pi_{tj_0} \times \Pi_{tj_1} > 0$, otherwise Π is deterministic and therefore periodic. We may suppose, without loss of generality, $j_0 = 0$, $j_1 = 1$. Now consider different possible values of t:

(a) $t = 0$ or 1: If for any l, $l \neq 0$, $l \neq 1$ we have $\Pi_{l0} = \Pi_{l1} = 0$, Π should be reducible. Let then l be such that either $\Pi_{l0} \neq 0$ or $\Pi_{l1} \neq 0$, $l \neq 0$, $l \neq 1$. Then, by taking s being either j_0 or j_1 and $u = t$ (20′) is satisfied.

(b) $t \geq 2$: We may take, without loss of generality, $t = 2$. Suppose, *ad absurdo*, the lemma false; then, necessarily, $\Pi_{u0} = \Pi_{u1} = 0$ for all $u \neq 2$, since on the contrary (20′) is satisfied with $s = 0$ or 1.

For any $l \neq 2$, let s_l be such that $\Pi_{l,s_l} > 0$; then the states $s_3, ..., s_{k-1}$ are necessarily distinct, otherwise (20′) is satisfied; so

$$\{s_3, ..., s_{k-1}\} = \{2, ..., k-1\}/\{r\}$$

for some r, as illustrated in the following:

0	1	s_2	s_1	s_{k-1}
0	0	....	1	
0	0	1	....	
1	1			
0	0	....	....	....
.	.	....	....	....
.	.			
.	.			
0	0	....	....	1 ...

This also implies that $s_0 = s_1 = i$ for some $i \geq 2$ and $r = i$, otherwise the lemma is true. Now r could not be equal to 2, otherwise either the lemma is true or we must have $\Pi_{ij} = 0$ for $i \geq 3$, $j \leq 2$, which implies that Π is reducible. We consider the case $r \geq 3$.

It is now clear that $\Pi_{l,k} = 0$ for any $k \neq s_l$ and any $l \neq 2$, and $\Pi_{2,k} = 0$ for any $k \geq 2$. Therefore we have

$$\Pi_{03} \times \Pi_{13} \times \Pi_{3s_3} \times \cdots \times \Pi_{k-1, s_{k-1}} > 0$$

This implies that Π is periodic, a contradiction, and achieves the proof of the lemma.

Let us now consider the case $k = 2$.

We look for solutions of (15) such that all the parameters, which are in this case $z_{01}(\tau)$, but two, are vanishing. The nonvanishing parameters are denoted $z_{01}(\tau)$, $z_{01}(\sigma)$, $\tau \neq \sigma$, $|\sigma| = |\tau|$, and they satisfy $z_{01}(\tau) + z_{01}(\sigma) = 0$. A similar discussion as above leads to an interval $[0, \alpha]$, $\alpha > 0$, for $z_{01}(\tau)$ and another one $[\beta, 0]$, $\beta < 0$, for $z_{01}(\sigma)$, if the following conditions are satisfied:

$$\mu_\pi(0, \tau, 0)\, \mu_\pi(1, \tau, 1) > 0 \tag{22}$$

$$\mu_\pi(1, \tau, 0)\, \mu_\pi(0, \sigma, 1) > 0 \tag{23}$$

We may suppose, without loss of generality, that $\Pi_{00} \Pi_{01} \Pi_{10} > 0$, a condition which is satisfied if Π is aperiodic and irreducible. By repeating similar arguments as above, it turns out that this condition on Π_{ij} allows us to find σ and τ, $\sigma \neq \tau$, $|\sigma| = |\tau|$, satisfying (22) and (23). This achieves the proof of the theorem.

2. TIME REVERSIBILITY

A stationary probability measure ν on $(\Omega_A, \mathscr{A}_A, \sigma)$ is time reversible if

$$\nu(\omega_0 = x_0, \ldots, \omega_n = x_n) = \nu(\omega_0 = x_n, \omega_1 = x_{n-1}, \ldots, \omega_n = x_0) \tag{24}$$

for any n and any $(x_0, \ldots, x_n)$. If $\nu = \mu_\pi$, this property is equivalent to

$$p_i \Pi_{ij} = p_j \Pi_{ji}$$

We denote by $R_{\pi, A}$ the subset of all time-reversible probability measures of $C_{\pi, A}$. For simplicity, we omit the index A in what follows. We shall construct measures in R_π distinct from μ_π. For this purpose, we use the above construction. Denote by $R_{\pi,n}$ the set of measures in $C_{\pi,n}$ which are time reversible in the sense of (24). We have the following lemma.

Lemma 2. Let $\mu \in R_{\pi,n}$; then $\varphi_n \mu$, given by (10), is in R_π.

Proof. One obtains, by straightforward computation, for $l = 0, 1, \ldots, n-1$, $p \in \mathbb{N}^*$, and $x \in K^{\mathbb{Z}}$ the following equations:

(i) For $\mu \in C_{\pi,n}$

$$\begin{aligned}
&\nu_0(\{\omega : \omega_0 = x_0, \ldots, \omega_{pn} = x_{pn}\}) \\
&\quad = \mu(x_{(p-1)n}, \ldots, x_{pn}) \\
&\qquad \times \mu(\omega_0 = x_{(p-2)n}, \ldots, \omega_{n-1} = x_{(p-1)n-1} \mid \omega_n = x_{(p-1)n}) \\
&\qquad \times \cdots \times \mu(\omega_0 = x_0, \ldots, \omega_{n-1} = x_{n-1} \mid \omega_n = x_n)
\end{aligned}$$

(ii) For $\mu \in \mathscr{R}_{\pi,n}$

$$\nu_0(\{\omega : \omega_0 = x_{pn}, \ldots, \omega_{pn} = x_0\}) = \nu_0(\{\omega : \omega_0 = x_0, \ldots, \omega_{pn} = x_{pn}\})$$

(iii)

$$\sigma^l \nu_0(\{\omega\colon \omega_0 = x_{pn},..., \omega_{pn} = x_0\}) = \sigma^{n-l}\nu_0\{(\omega\colon \omega_0 = x_0,..., \omega_{pn} = x_{pn}\})$$

The lemma follows from these equations and from the definition of $\varphi_n \mu$.

Theorem 2. If Π is reversible, irreducible, and aperiodic, then for all $n > N$, N sufficiently great, $R_{\pi,n,A}$ contains measures distinct from μ_π.

Proof. In order to construct $\mu \in R_\pi$ for Π irreducible and aperiodic which is distinct from μ_π we proceed as in Section 1, by solving (11)–(13) for any τ. On account of the reversibility (24) this system is equivalent to (11), (13), and (24). The linear system (14) is simplified to the first k equations. So, for any fixed τ, we have to solve $AZ = 0$ for $Z = (z_{i,j})$ with matrix A being the k first lines of (14). The rank of this system is k. We take as principal variables those indexed by $(i, 0)$ for all i. They are expressed in terms of the other variables, the parameters J, as follows:

$$z_{i0} = -\sum_{j=1}^{k-1} z_{ij}, \qquad \forall i = 0,..., k-1$$

Let us denote $r(\tau_1,..., \tau_n) = (\tau_n,..., \tau_1)$. We look for a solution such that for some τ and $\sigma \neq \tau$, $\tau \neq r(\tau)$, and $\sigma \neq r(\sigma)$, $z_{i_0,j_0}(\tau)$, $z_{i_0,j_1}(\tau)$, $z_{i_0,j_0}(\sigma)$, and $z_{i_0,j_1}(\sigma)$ are nonvanishing, with $j_0 \neq 0$, $j_1 \neq 0$, $j_0 \neq j_1$, and

$$\sum_\tau z_{ij}(\tau) = 0$$

In order to ensure the reversibility of μ, we have to construct solutions such that

$$z_{ij}(\tau_1,..., \tau_n) = z_{ji}(\tau_n,..., \tau_1)$$

On account of the reversibility of Π, we see that $\Pi_{ij} > 0$ iff $\Pi_{ji} > 0$.

Then $z_{j_0,i_0}(r(\tau))$, $z_{j_1,i_0}(r(\tau))$, $z_{j_0,i_0}(r(\sigma))$, and $z_{j_1 i_0}(r(\sigma))$ are also nonvanishing. We take all the others parameters vanishing.

Here the positivity conditions are

$$z_{i_0,j_0}(\tau) \geqslant -\mu_\pi(i_0 \tau j_0)$$

$$z_{i_0,j_1}(\tau) \geqslant -\mu_\pi(i_0 \tau j_1)$$

$$z_{i_0,j_0}(\tau) + z_{i_0,j_1}(\tau) \leqslant \mu_\pi(i_0 \tau 0)$$

The same inequalities hold for σ. As above, it is sufficient for this to have

$$\mu_\pi(i_0 \tau j_0) > 0, \qquad \mu_\pi(i_0 \sigma j_1) > 0$$

They are satisfied if there exist $s, t \in K$, $s \neq t$, such that

$$\Pi_{i_0 s} \Pi_{t j_0} \Pi_{t j_1} > 0$$

This is satisfied for any aperiodic matrix Π.

The proof goes through as above. ▮

Remark. In the case of a strictly positive matrix Π, the set of $\mu \in R_{\pi,n}$ which coincide with μ_π on the first n coordinates contains a nonempty convex open set.

In fact, to construct reversible invariant measures on K^{n+1} distinct from μ_π, we proceed in solving the linear system

$$\sum_{x_n} \mu(x_0, ..., x_n) = \mu_\pi(x_0, ..., x_{n-1}) \tag{25}$$

$$\sum_{x_1, ..., x_{n-1}} \mu(i, x_0, ..., x_{n-1}, j) = \mu_\pi(\omega_0 = i, \omega_n = j) \tag{26}$$

$$\mu(x_0, ..., x_n) = \mu(x_n, ..., x_0) \tag{27}$$

Using (27), we are left with the system (25) and (26) and a reduced number of unknowns $\mu(x_0, ..., x_n)$. Let us denote by $I(r)$ the subset of K^{n+1} such that $x = r(x)$. When $x = r(x)$, we choose one of $\mu(x)$ and $\mu(r(x))$ as unknown, eliminating the other in view of (27). Thus we have a new system given by (25) and (26) and a reduced set of unknowns, which we denote $AX = Y^0$. Denote by X^0 the solution $\mu_\pi(x_0, ..., x_n)$. A general solution $X = X^0 + Z$, $Z \in \mathrm{Ker}(A)$, has to satisfy the positivity of μ, i.e., $Z \geqslant -X^0$. To compute the number of unknowns, we distinguish two cases:

(a) If $n = 2p + 1$: The number of unknowns is $\frac{1}{2}k^{p+1}(k^{p+1} + 1)$.

(b) If $n = 2p$: The number of unknowns is $\frac{1}{2}k^{p+1}(k^p + 1)$.

Now, in the system (25) and (26) some equations are redundant. In fact, if $(x_0, ..., x_{n-1}) = (x_{n-1}, ..., x_0)$, then Eq. (25) implies, in view of the reversibility of μ_π,

$$\sum_y \mu(x_{n-1}, ..., x_0, y) = \mu_\pi(x_{n-1}, ..., x_0)$$

Similarly, in (26) we may take only $i \leqslant j$. Thus the number of equations is reduced:

(a) If $n=2p+1$: The number of unknowns is $k^{(2p+1)}-k^{(p+1)}+\frac{1}{2}k(k+1)$.

(b) If $n=2p$: The number of unknowns is $k^{2p}-k^{p}+\frac{1}{2}k(k+1)$.

Therefore:

(a) If $n=2p+1$:

$$\dim \operatorname{Ker} A \geqslant \frac{1}{2} k^{p+1}(k^{p+1}+1)-k^{p+1}(k^{p}-1)-\frac{k}{2}(k+1)$$

(b) If $n=2p$:

$$\dim \operatorname{Ker} A \geqslant \frac{1}{2} k^{p+1}(k^{p}+1)-k^{p}(k^{p}-1)-\frac{k}{2}(k+1)$$

This is greater than 1 for any $n \geqslant 2$ and $k \geqslant 2$.

This shows that the kernel of A is a nontrivial vector space. Thus, $\operatorname{Ker}(A) \cap \{Z: Z \geqslant -X^0\}$ is a convex neighborhood of 0 in $\operatorname{Ker}(A)$.

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